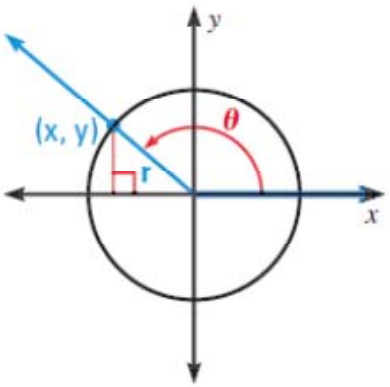
13.3 Evaluate Trigonometric Functions of Any Angle

KEY CONCEPT *For Your Notebook*

General Definitions of Trigonometric Functions

Let θ be an angle in standard position, and let (x, y) be the point where the terminal side of θ intersects the circle $x^2 + y^2 = r^2$. The six trigonometric functions of θ are defined as follows:



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$

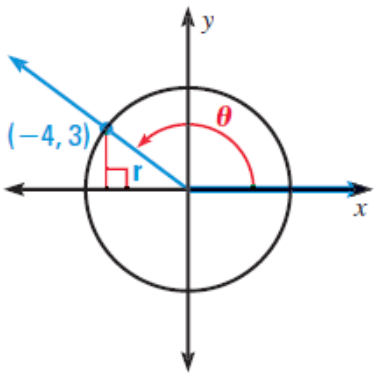
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

These functions are sometimes called *circular functions*.

EXAMPLE 1 Evaluate trigonometric functions given a point

Let $(-4, 3)$ be a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.



Solution

Use the Pythagorean theorem to find the value of r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$$

Using $x = -4$, $y = 3$, and $r = 5$, you can write the following:

$$\sin\theta = \frac{y}{r} = \frac{3}{5} \qquad \cos\theta = \frac{x}{r} = -\frac{4}{5} \qquad \tan\theta = \frac{y}{x} = -\frac{3}{4}$$

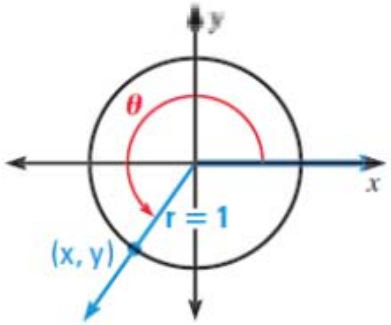
$$\csc\theta = \frac{r}{y} = \frac{5}{3} \qquad \sec\theta = \frac{r}{x} = -\frac{5}{4} \qquad \cot\theta = \frac{x}{y} = -\frac{4}{3}$$

KEY CONCEPT *For Your Notebook*

The Unit Circle

The circle $x^2 + y^2 = 1$, which has center $(0, 0)$ and radius 1, is called the **unit circle**. The values of $\sin\theta$ and $\cos\theta$ are simply the y-coordinate and x-coordinate, respectively, of the point where the terminal side of θ intersects the unit circle.

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

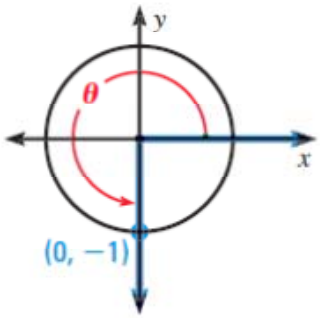


EXAMPLE 2 Use the unit circle

Use the unit circle to evaluate the six trigonometric functions of $\theta = 270°$.

Solution

Draw the unit circle, then draw the angle $\theta = 270°$ in standard position. The terminal side of θ intersects the unit circle at $(0, -1)$, so use $x = 0$ and $y = -1$ to evaluate the trigonometric functions.



$$\sin\theta = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \csc\theta = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec\theta = \frac{r}{x} = \frac{1}{0} \text{ undefined}$$

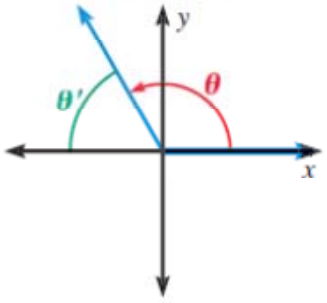
$$\tan\theta = \frac{y}{x} = \frac{-1}{0} \text{ undefined} \qquad \cot\theta = \frac{x}{y} = \frac{0}{-1} = 0$$

KEY CONCEPT *For Your Notebook*

Reference Angle Relationships

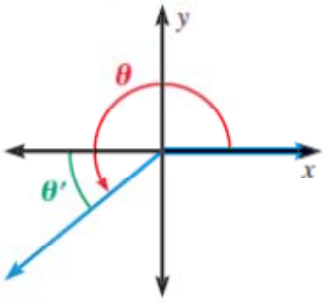
Let θ be an angle in standard position. The **reference angle** for θ is the acute angle θ' formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is shown below for nonquadrantal angles θ such that $90° < \theta < 360°$ $\left(\frac{\pi}{2} < \theta < 2\pi\right)$.

Quadrant II



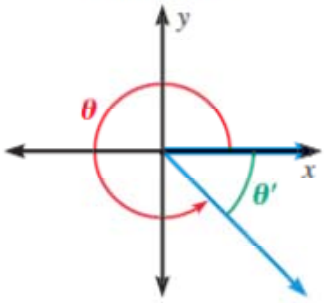
Degrees: $\theta' = 180° - \theta$
Radians: $\theta' = \pi - \theta$

Quadrant III



Degrees: $\theta' = \theta - 180°$
Radians: $\theta' = \theta - \pi$

Quadrant IV



Degrees: $\theta' = 360° - \theta$
Radians: $\theta' = 2\pi - \theta$

EXAMPLE 3 Find reference angles

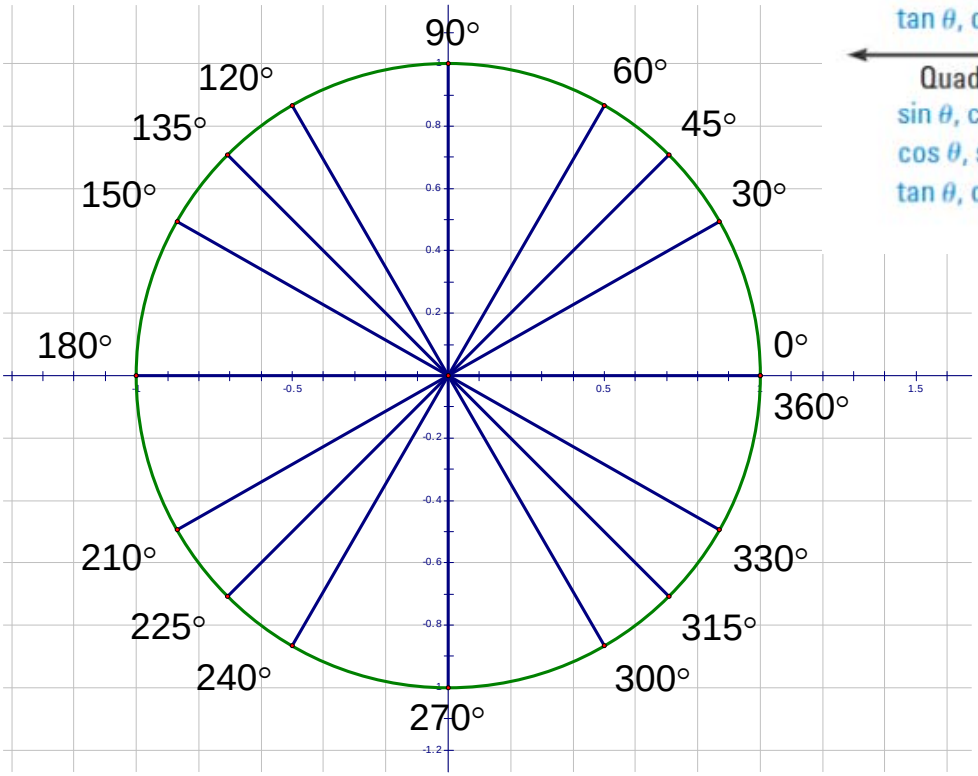
Find the reference angle θ' for (a) $\theta = \frac{5\pi}{3}$ and (b) $\theta = -130°$.

Solution

a. The terminal side of θ lies in Quadrant IV. So, $\theta' = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$.

b. Note that θ is coterminal with 230°, whose terminal side lies in Quadrant III. So, $\theta' = 230° - 180° = 50°$.

REFERENCE ANGLE RELATIONSHIPS



Signs of Function Values

Quadrant II	Quadrant I
$\sin\theta, \csc\theta$: +	$\sin\theta, \csc\theta$: +
$\cos\theta, \sec\theta$: −	$\cos\theta, \sec\theta$: +
$\tan\theta, \cot\theta$: −	$\tan\theta, \cot\theta$: +
Quadrant III	**Quadrant IV**
$\sin\theta, \csc\theta$: −	$\sin\theta, \csc\theta$: −
$\cos\theta, \sec\theta$: −	$\cos\theta, \sec\theta$: +
$\tan\theta, \cot\theta$: +	$\tan\theta, \cot\theta$: −

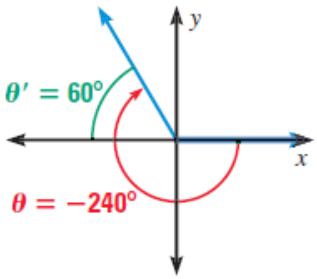
EXAMPLE 4 Use reference angles to evaluate functions

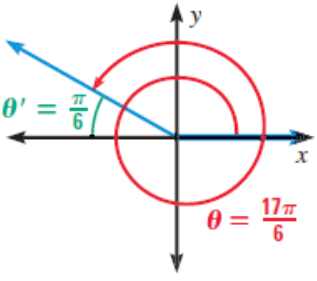
Evaluate (a) $\tan(-240°)$ and (b) $\csc\frac{17\pi}{6}$.

Solution

a. The angle $-240°$ is coterminal with $120°$. The reference angle is $\theta' = 180° - 120° = 60°$. The tangent function is negative in Quadrant II, so you can write:

$$\tan(-240°) = -\tan 60° = -\sqrt{3}$$



b. The angle $\frac{17\pi}{6}$ is coterminal with $\frac{5\pi}{6}$. The reference angle is $\theta' = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$. The cosecant function is positive in Quadrant II, so you can write:

$$\csc\frac{17\pi}{6} = \csc\frac{\pi}{6} = 2$$